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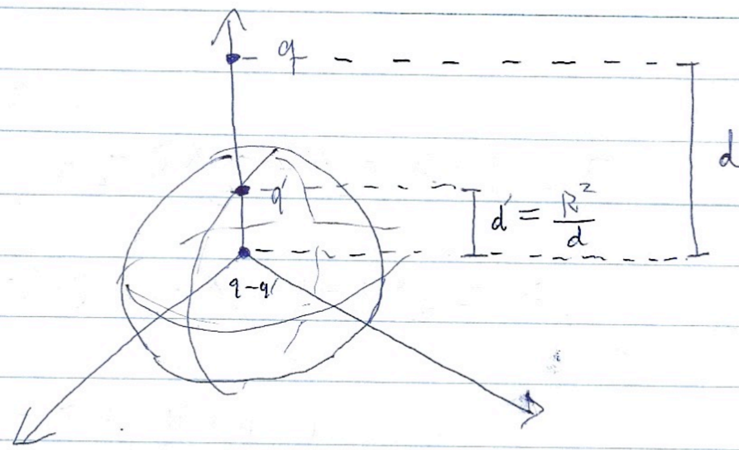
2.4. (a). We place the charge on the $\hat{z}$ axis. The image charge by symmetry is on $\hat{z}$ as well. It will have charge of $q' = -\frac{R}{d} q$, at distance $d' = \frac{R^2}{d}$ away from the origin.



The sphere already have charge of q' induced on its surface by the mirror charge. To spherically-symmetrically induce the other $Q - q'$ charges, we place a charge of $q - q'$ at the origin. The force induced on q will be (boundary condition).

$$|F| = K\frac{\left(\frac{R+d}{d} q\right) q}{d^2} + \frac{K\left(-\frac{R}{d} q\right) q}{\left(\frac{d^2 - R^2}{d}\right)^2} = 0$$

$$\left(\frac{R+d}{d^3}\right)\cancel{q} - \left(\frac{R}{\cancel{d}}\right)\cancel{q}\frac{d^{\cancel{2}}}{(d^2-R^2)^2} = 0$$

$$\frac{R+d}{d^3} = \frac{Rd}{(d^2-R^2)^2} = \frac{Rd}{(d-R)(d+R)(d-R)(d+R)}$$

$$R+d = \frac{Rd^4}{(d-R)(d+R)(d-R)(d+R)}$$

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$$\cancel{R}\left(\frac{d}{R}+1\right) = \frac{\cancel{R}\, d^4}{R^4\left(\frac{d}{R}-1\right)\left(\frac{d}{R}+1\right)\left(\frac{d}{R}-1\right)\left(\frac{d}{R}+1\right)}$$

$$\left(\frac{d}{R}-1\right)^2\left(\frac{d}{R}+1\right)^3 = \left(\frac{d}{R}\right)^4.$$ Let $x = \frac{d}{R}$.

$$(x-1)^2(x+1)^3 = x^4,$$

$$x^5 + x - 2x^3 + \cancel{x^4} + 1 - 2x^2 = \cancel{x^4}$$

$$\boxed{x^5 - 2x^3 - 2x^2 + x + 1 = 0}$$

This is a quartic, which can not be solved in general, relying numerical solutions, we look for solution of x that is greater than 1. The only solution is $\boxed{x = 1.6180339\ldots}$.

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2.4 (b) The force due to the two mirror charges is

$$F = kq^2 \left[\frac{R+d}{d^3} - \frac{Rd}{(d^2-R^2)^2} \right]$$

$a = d-R$, $d+R = a+2R$, $d = a+R$, so

$$F = kq^2 \left[\frac{(a+2R)}{(a+R)^3} - \frac{R(a+R)}{(a+2R)^2 a^2} \right]$$

$$= kq^2 \left[\frac{R(\frac{a}{R}+2)}{R^3(\frac{a}{R}+1)^3} - \frac{R^2(\frac{a}{R}+1)}{R^2(\frac{a}{R}+2)^2 a^2} \right].$$

$$= kq^2 \left[\frac{1}{R^2} \frac{(\frac{a}{R}+2)}{(\frac{a}{R}+1)^3} - \frac{1}{a^2} \frac{(\frac{a}{R}+1)}{(\frac{a}{R}+2)^2} \right].$$

In the limit of $a << R$, the term $\frac{1}{a^2}$ will dominate $\frac{1}{R^2}$, Thus we consider the second term.
In the limit $a << R$, $\frac{a}{R} \to 0$. Thus $\frac{(\frac{a}{R}+1)}{(\frac{a}{R}+2)^2} \to \frac{1}{4}$.

$$\lim_{a<<R} F = kq^2 \left[-\frac{1}{a^2} \frac{1}{4} \right].$$

$$\boxed{= \frac{q^2}{16\pi\varepsilon_0} \left[-\frac{1}{a^2} \right]}$$

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